

# Simple experimental methods for trapped ion quantum processors

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Two techniques are described that simplify the experimental requirements for measuring and manipulating quantum information stored in trapped ions. The first is a new technique using electron shelving to measure the populations of the Zeeman sublevels of the ground state, in an ion for which no cycling transition exists from any of these sublevels. The second technique is laser cooling to the vibrational ground state, without the need for a trap operating in the Lamb-Dicke limit. This requires sideband cooling in a sub-recoil regime. We present a thorough analysis of sideband cooling on one or a pair of sidebands simultaneously.

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Laser cooling [1–3] and electron shelving detection techniques [4–6] were developed in the pursuit of better control of a basic physical system, and high precision experiments. Up to now such experiments have concentrated on atomic transitions that offer good prospects as frequency standards, or that allow sensitive detection of some physical effect. The relatively new subject of quantum information theory [7–9] has led to interest in a different experimental approach. Here, an important aim is to realise the preparation, coherent control, and detailed measurement of some sufficiently complex physical system. This allows one to realise simple networks of elementary quantum operations (quantum gates), and to test basic ideas in quantum information theory, such as data compression, quantum error correction, and simple quantum algorithms. The new feature is that such experiments do not require high precision as an end in itself (unlike frequency standards), although high precision is of course very desirable, nor do they require sensitivity to some physical effect. Instead, one’s attention is focused purely on the logical properties of the state transformations and measurements that can be carried out.

The significance of experiments on trapped ions to quantum information physics was recognised by Cirac and Zoller [10], who proposed the use of a linear ion trap to realise the essential elements of a quantum computer. Although a real ion trap will not permit large-scale quantum computing [11–13], such a system should allow experiments on a few tens of qubits, and is one of the most promising for such a purpose. Currently a few groups worldwide are developing experiments to pursue such ideas [13].

In this paper we consider the question “what is the simplest method to build and operate a linear ion trap quantum information processor?” One of the available options is to use an ion such as calcium or strontium whose ground state has total spin half. The use of the two Zeeman sublevels of the ground state of each ion naturally suggests itself as a means to store each qubit, but currently this option has not received much attention, partly because the method to measure the final state, the “readout” in computer terminology, is not obvious. We propose in sections 1 and 2 a simple electron shelving technique that will permit such measurements. Our method does not realise an ideal von Neumann measurement (projection onto an orthonormal basis) but nevertheless permits efficient quantum computing and quantum state tomography. This is discussed in section 3.

In section 4 we consider state preparation in this system. The most difficult part of the state preparation is the cooling of the motional degrees of freedom. We present the results of a study of sideband cooling in the regime where the confinement is of intermediate tightness (Lamb-Dicke parameter of order one), since this is typically the regime in which one would wish to operate the trap as a “processor”. We find that first sideband cooling operates well for Lamb-Dicke parameters up to  $\eta \sim 0.6$ , and indeed sub-recoil cooling is possible. By combining excitation of the first sideband with a higher one, cooling to the ground state can be achieved for Lamb-Dicke parameters up to  $\eta \sim 3$ . We find that the best choice of sideband is higher than one might expect, and the analysis reveals the optimum ratio of laser intensities on the two sidebands.

## I. CHOICE OF TRANSITION

Consider a linear ion trap, in which the vibrational frequencies along the three axes, for a single confined ion, are  $\omega_x \simeq \omega_y \gg \omega_z$ . Each ion must store one qubit (or possibly more than one), so we require two long-lived states in the ion, between which transitions can be driven coherently. The recoil energy is defined  $E_R \equiv (\hbar k)^2/(2M)$  where  $k$  is the wavevector of the radiation used to drive the transition and  $M$  is the mass of a single ion. The Lamb-Dicke parameter is

$$\eta \equiv m \cos(\theta) (E_R/\hbar\omega_z)^{1/2} \quad (1)$$

where  $\theta$  is the angle between the laser beam(s) and the  $z$  axis ( $\mathbf{k} \cdot \mathbf{z} \equiv k \cos(\theta)$ ) and  $m = 1$  for single photon transitions,  $m = 2$  for Raman transitions, assuming the geometry  $\mathbf{k}_1 \cdot \mathbf{z} = -\mathbf{k}_2 \cdot \mathbf{z}$ .

We require the Lamb Dicke parameter not to be vanishingly small, in order that the radiation can affect the motional state of the ions, and therefore optical rather than radio-frequency radiation must be used. Since the separation of adjacent trapped ions is in the region 10 to 100  $\mu\text{m}$ , radiation of sub-micron wavelength is also preferable in order to address individual ions. These considerations lead to two approaches to information processing experiments: either a single-photon transition between metastable states separated by optical frequencies is adopted, or an rf transition is driven using optical radiation by means of the stimulated Raman effect.

We will consider the case of Raman transitions, since the laser frequency stability requirement is much less restrictive [14]. The Raman transition is sensitive only to the difference of the frequency of the two laser beams used to drive it. If both beams are derived from the same laser, the laser's frequency fluctuations cancel out, and we only have to concern ourselves (to first order) with power fluctuations. A single-photon transition, by contrast, would require a laser system close to the state of the art in frequency stability.

Atomic energy levels that have long natural lifetimes and separations in the rf regime almost always owe their energy separation to the hyperfine interaction, or the Zeeman or Stark effects. The singly-charged  $^9\text{Be}$  ion is a good candidate for information processing purposes because it has hyperfine structure. However, the transition wavelengths in ions such as  $^{40}\text{Ca}$  and  $^{88}\text{Sr}$  are more easily accessible with current laser technology. This leads us to consider these ions as candidates for quantum information experiments. Since they lack hyperfine structure, and we wish to use long-lived states with rf separation, we store each qubit in the two-dimensional Hilbert space spanned by the two Zeeman sublevels  $|M = \pm 1/2\rangle_g$  of the ground state. The original proposal of Cirac and Zoller [10] made use of a third long-lived state in each ion in order to bring about quantum gates such as XOR, which is not available here. However, we can avoid the need for such a state either by use of the “magic Lamb-Dicke parameter” method of Monroe *et al.* [15], or by using the first excited state of the second normal mode of oscillation.

One disadvantage of our choice is that the transition  $|M = -1/2\rangle_g \leftrightarrow |M = +1/2\rangle_g$  is sensitive to the local magnetic field to first order. Although this would rule out such a transition as a frequency standard, it does not for information processing. The reason is that we only require short term stability, not absolute precision, in the energy level separations. The actual value of these separations is totally immaterial, as long as the Raman resonance condition can be found precisely in any given experiment. An advantage of our choice is that there are no further sublevels in the ground state into which population can “leak”, which is one source of error in quantum information processing.

A major apparent problem with our choice is in the final state detection. In quantum processing, we require a sensitive measurement of the final state of each individual qubit. Sufficient signal to noise ratio to make such a measurement is quite rare in physics, but it can be done for individual trapped atoms by exciting fluorescence on an optical transition that is resonant for one state (say  $|M = +1/2\rangle_g$ ) but not for the orthogonal state. Typically at least a few thousand cycles of excitation followed by spontaneous decay must be completed in order to get an unambiguous signal. This is possible if the optically excited state can only decay to the  $|M = +1/2\rangle_g$  ground state, that is not to  $|M = -1/2\rangle_g$  and not to some other metastable state. However, in ions such as Ca, Sr, Ba and Hg all the short-lived excited states (one of which must be used for efficient fluorescence detection) have a chance of decaying to a metastable state (see fig. 1), causing fluorescence to stop after about 3 to 10 cycles. This is the problem we wish to address.

## II. ELECTRON SHELVING ZEEMAN STATE MEASUREMENT

Our scheme to detect the final Zeeman ground state of a single ion of Ca, Sr, or a similar candidate, requires three laser wavelengths, exciting the transitions  $S_{1/2} \leftrightarrow P_{3/2}$ ,  $S_{1/2} \leftrightarrow P_{1/2}$ , and  $D_{3/2} \leftrightarrow P_{1/2}$  (see fig. 1). Since the latter two are required for laser cooling in any case, the main cost of the scheme is one additional laser wavelength. The wavelengths for  $^{40}\text{Ca}$  are 394 nm, 397 nm and 866 nm respectively. We will use these values as a shorthand to refer to the transitions, although obviously the method works essentially unchanged for all similar ions.

A magnetic field is imposed in a direction along the wavevector of the the 394 nm radiation, which typically is not the  $z$  axis of the ion trap. We adopt the magnetic field direction as the quantization axis for the purpose of defining spin states and polarizations. The 394 nm radiation is polarized  $\sigma^+$ . The polarization of the other beams is almost unconstrained, linear polarization at right angles to the magnetic field direction (i.e. equal amounts of  $\sigma^+$ ,  $\sigma^-$ ) will do.

During the first part of the detection scheme, the magnetic field must be large enough to enable the Zeeman components of the  $S_{1/2} \leftrightarrow P_{3/2}$  transition (394 nm) to be resolved. In other words, their splitting is of the order of or greater than the inverse of the lifetime of the excited state ( $P_{3/2}$ ). The 394 nm radiation is tuned to resonance with

the  $S_{1/2}, M = +1/2 \leftrightarrow P_{3/2}, M = 3/2$  transition, and therefore excites transitions from  $|M = +1/2\rangle_g$  much more strongly than from  $|M = -1/2\rangle_g$ .

Let  $\Pi_{\pm}$  be the initial populations of the ground state sublevels, i.e. the diagonal elements of the single-ion density matrix in the  $|M = \pm 1/2\rangle_g$  basis.

The measurement proceeds in two steps: shelving followed by fluorescence. First the 394 nm radiation is pulsed on for a time sufficient to cause about 10 cycles of excitation and spontaneous decay. After this time the population that was originally in  $|M = +1/2\rangle_g$  has been optically pumped (shelved) into the  $D$  states, most of it (about 90%) in the  $D_{5/2}$  state, while the population in  $|M = -1/2\rangle_g$  is almost unaffected. For a single ion experiment, this means that the state jumps abruptly to the  $D$  manifold or to  $|M = -1/2\rangle_g$  with probabilities proportional to  $\Pi_+$ ,  $\Pi_-$  respectively. Next, the 394 nm radiation is turned off, the magnetic field is reduced, and the two other wavelengths are turned on. Fluorescence is detected during a time equal to a few percent of the lifetime of the  $D_{5/2}$  state. Since this time is more than  $10^7$  times longer than that of the  $P_{1/2}$  state, of the order of a million cycles of fluorescence can take place, making a large accumulated signal. The probability of observing fluorescence is proportional approximately to  $\Pi_- + 0.1\Pi_+$ . A further signal for normalisation or checking purposes can be obtained after the  $D_{5/2}$  state decays, if the ion state jumped there during the shelving pulse.

We have analysed this scheme by numerically solving the complete set of rate equations, including the 18 Zeeman sublevels and 40 allowed electric dipole transitions in the problem. This enables us to calculate the requirements on magnetic field strength and laser pulse timing, and to interpret the signal precisely. A rate equation rather than optical Bloch equation approach is sufficient since during the shelving pulse, optical coherences play no role in determining populations, and during the fluorescence detection we impose a magnetic field sufficient to prevent accumulation of population in a “dark state”, that is, a spin state in the  $D_{3/2}$  Zeeman manifold that does not couple to the 866 nm radiation. For this, the dark state must Larmor precess to a bright state much faster than the optical pumping rate among the Zeeman sublevels of the  $D_{3/2}$  state. The calculation tells us this optical pumping rate.

In order to write the equations, let us number the five energy levels in order of increasing energy, as in fig. 1. Let  $N_M^{(n)}$  be the population of the  $M$  Zeeman sublevel of the  $n$ th level. The total rate of change of population  $N_M^{(n)}$  is  $(dN_M^{(n)}/dt)_{\text{sp}} + (dN_M^{(n)}/dt)_{\text{dr}}$ , a sum of spontaneous and driven terms. The spontaneous feeding and decay terms are

$$\left(\frac{dN_M^{(n)}}{dt}\right)_{\text{sp}} = \sum_{n' > n} \Gamma_{nn'} \sum_{q=-1}^1 \left|C_{J(n),M}^q\right|^2 N_{M+q}^{(n')} - \sum_{n' < n} \Gamma_{n'n} N_M^{(n)}, \quad (2)$$

where  $C_{J(n),M}^q$  are the Clebsch-Gordan coefficients.

The driven terms are optical excitations caused by the radiation field, taking into account the Rabi frequency  $\Omega$ , detuning  $\delta$  and the polarisation of the radiation. For example, during the initial “shelving” stage when only  $\sigma^+$  polarized 394 nm radiation is present, tuned to resonance with the Zeeman-shifted  $S_{1/2}, M = 1/2 \leftrightarrow P_{3/2}, M = 3/2$  transition, the driven terms are

$$\left(\frac{dN_{1/2}^{(1)}}{dt}\right)_{\text{dr}} = -\left(\frac{dN_{3/2}^{(5)}}{dt}\right)_{\text{dr}} = -\left(N_{1/2}^{(1)} - N_{3/2}^{(5)}\right) \frac{\pi}{2} \Omega_{15}^2 g_5(0), \quad (3)$$

$$\left(\frac{dN_{-1/2}^{(1)}}{dt}\right)_{\text{dr}} = -\left(\frac{dN_{1/2}^{(5)}}{dt}\right)_{\text{dr}} - \left(N_{-1/2}^{(1)} - N_{1/2}^{(5)}\right) \frac{\pi}{2} \frac{1}{3} \Omega_{15}^2 g_5\left(\frac{2}{3}\mu_B B/\hbar\right), \quad (4)$$

where the lineshape function is defined by

$$g_i(\delta) = \frac{\Gamma_i/2\pi}{\delta^2 + \Gamma_i^2/4}, \quad (5)$$

$$\Gamma_i = \sum_j \Gamma_{ji}. \quad (6)$$

We obtain the spontaneous decay rates  $\Gamma_{ji}$  from [16]. For Ca, they are  $\Gamma_{14} = 2\pi \times 20.7$  MHz,  $\Gamma_{24} = 2\pi \times 1.69$  MHz,  $\Gamma_{15} = 2\pi \times 21.5$  MHz,  $\Gamma_{25} = 2\pi \times 0.177$  MHz,  $\Gamma_{35} = 2\pi \times 1.58$  MHz. The Clebsch-Gordan coefficients and Landé  $g$  factors are obtained by calculation in the  $LS$  coupling approximation. Note that we have neglected the far-off-resonant excitation rates in the rate equations written above. They were in fact included in the numerical calculations, though their influence is negligible.

We can quantify the success of the spin measurement by two parameters: the ability to discriminate the  $|M = \pm 1/2\rangle_g$  states, and the size of the signal. We parametrize the former by calculating a “shelving efficiency”  $\epsilon$  defined to be  $\epsilon = S_+ - S_-$ , where  $S_+ = \sum_M N_M^{(3)}$  is the population in the  $D_{5/2}$  levels after the shelving pulse, when the population is initially all in  $|M = +1/2\rangle_g$  (i.e.  $\Pi_+ = 1$ ), and  $S_-$  is the same quantity when the population is initially all in  $|M = -1/2\rangle_g$  (so  $\Pi_- = 1$ ). In fig. 2 we show  $\epsilon$  as a function of the magnetic field  $B$ , for two values of the Rabi frequency,  $\Omega_{15} = \Gamma_{15}$  (dashed line) and  $\Omega_{15} = \Gamma_{15}/4$  (full line), with the shelving pulse duration optimized for maximum  $\epsilon$  at each value of  $B$ . At large fields,  $\epsilon$  tends to  $\Gamma_{35}/(\Gamma_{35} + \Gamma_{25}) \simeq 0.899$ . The field required to obtain  $\epsilon > 0.5$  is of order  $B \simeq 3 \times 10^{-3}$  T, which is easily produced in the lab.

It is important to get the shelving pulse duration right, since if it is too long then all the ground state population will be pumped to the  $D$  states, irrespective of the initial values of  $\Pi_{\pm}$ . This duration, and the required magnetic field, can be estimated by a reduced set of rate equations, in the limit that all rates are fast compared to two, namely the rate  $R_1$  of transfer from  $|M = -1/2\rangle_g$  to  $|M = +1/2\rangle_g$ , and the rate  $R_2$  of transfer from  $|M = +1/2\rangle_g$  to the  $D$  states. These are

$$R_1 = \frac{2}{3} \frac{\pi}{2} \frac{1}{3} \Omega_{15}^2 g_5 \left( \frac{2}{3} \mu_B B / \hbar \right), \quad (7)$$

$$R_2 = \frac{\Omega_{15}^2}{2\Omega_{15}^2 + \Gamma_5^2} \Gamma_{35}. \quad (8)$$

The reduced set of equations are the following:

$$\frac{dN_{-1/2}^{(1)}}{dt} = -R_1 N_{-1/2}^{(1)}, \quad (9)$$

$$\frac{dN_{1/2}^{(1)}}{dt} = R_1 N_{-1/2}^{(1)} - R_2 N_{1/2}^{(1)}, \quad (10)$$

$$\frac{dN^{(D)}}{dt} = R_2 N_{1/2}^{(1)}, \quad (11)$$

where the final equation accounts for all population in the  $D$  levels. Writing  $S_+ = 0.899 N^{(D)}(t)$  when  $N_{1/2}^{(1)}(0) = 1$ ,  $S_- = 0.899 N^{(D)}(t)$  when  $N_{-1/2}^{(1)}(0) = 1$ , we obtain the solution

$$\epsilon(t) = 0.899 \frac{R_2}{R_2 - R_1} (e^{-R_1 t} - e^{-R_2 t}). \quad (12)$$

This solution reproduces the result of the complete calculation quite well, see fig. 3. The optimum pulse length, when  $d\epsilon/dt = 0$ , is therefore

$$t_{\max} = \frac{\ln(R_2/R_1)}{R_2 - R_1}, \quad (13)$$

and the requirement on the magnetic field is  $R_2 \gg R_1$  which is roughly equivalent to the requirement that the Zeeman splitting is large compared to  $\Gamma_5$ , as one would expect. The typical order of magnitude for  $t_{\max}$  is  $10/\Gamma_{35} \simeq 1 \mu\text{s}$ .

To complete the analysis, we also solved the complete set of rate equations during the detection phase, when laser radiation at 397 nm and 866 nm is present (polarised perpendicular to the magnetic field direction). The observed fluorescence rate will be proportional to the population of the  $P_{1/2}$  level,  $N_{-1/2}^{(4)} + N_{+1/2}^{(4)}$ . We show in fig. 4 the steady state value of this population as a function of magnetic field, with the lasers always tuned to the zero-field resonance frequencies, and initial condition  $\Pi_- = 1$ . Comparing fig. 4 with fig. 2, the conclusion is that it is not absolutely necessary to reduce the magnetic field between the shelving and detection phases, as long as sufficient laser power is used during the detection phase, but that a reduction to  $10^{-3}$  T is useful to increase the observed signal. Since the  $D_{5/2}$  state lifetime is long, there is plenty of time to reduce the magnetic field if so desired.

We find also that the steady-state  $P_{1/2}$  population remains of order 1/4 when the magnetic field is large enough to produce Larmor precession much faster than the optical pumping rate among the Zeeman sublevels of  $D_{3/2}$ . This is made possible by the favourable condition  $\Gamma_{14} \gg \Gamma_{24}$ , and is necessary in order to avoid population accumulating in a dark state of  $D_{3/2}$ .

### III. REQUIREMENTS ON MEASUREMENTS FOR QUANTUM COMPUTING

The scheme we have discussed does not achieve an ideal measurement, which would be represented by the value  $\epsilon = 1$ . In this section we will discuss whether this is a problem.

First consider measuring a single ion. Any  $\epsilon > 0$  will allow the populations  $\Pi_{\pm}$  to be deduced by repeating the measurement many times (preparing the same state each time). It is sufficient that the graph of expected signal (the mean number of times fluorescence is seen during the detection phase) versus  $\Pi_+$  is single-valued, since then  $\Pi_+$  (and therefore  $\Pi_-$ ) can be deduced unambiguously from the mean signal. If  $\epsilon$  is small the measurement will need to be repeated more times in order to get good statistics to estimate  $\Pi_+$ , but our value  $\epsilon \simeq 0.9$  is not significantly worse than  $\epsilon = 1$  in this respect. Methods available for other ions, such as probing a cycling transition in Be, have  $\epsilon$  closer to 1, but the signal to noise ratio is worse because imperfection in the laser polarisation limits the number of fluorescence cycles per measurement to a few thousand, instead of about a million as in our method.

Now consider a many-ion quantum processor containing  $N$  ions. It is clear that  $\epsilon \sim 0.9$  is sufficient for information processors of small or intermediate size, simply because  $\epsilon^N$  remains non-negligible (it falls to 0.01 at  $N = 44$ ). The question is whether such non-ideal measurements remain useful for a large quantum computer, containing possibly thousands of qubits. Although ion traps may not offer the best technology to build large quantum computers in the future, the question of measurement errors will remain significant. We will discuss this using the language of ion traps; the results can be translated directly to other systems.

In the context of quantum computation, the final state of the computer represents the output of some algorithm. First consider the case that this output (before measurement) is a product state where each ion is either in the state  $|M = -1/2\rangle_g$  or  $|M = +1/2\rangle_g$ . The probability of obtaining a correct readout after a single run of the algorithm is  $\epsilon^N$  where  $N$  is the number of ions in  $|M = +1/2\rangle_g$ . It might be imagined that this exponential dependence on  $N$  renders the complete algorithm (including final measurement) computationally inefficient. In fact this is not the case. On repeating the whole algorithm and final measurement  $r$  times, independent statistics are gathered on the final state of each ion, with the aim of deciding whether the mean number of times fluorescence is obtained is equal to  $r$  or  $r(1 - \epsilon)$ . The probability that this mean result is interpreted correctly is of order  $[1 - (1 - \epsilon)^r]^N$ . The number of repetitions required to make the overall success probability close to one is therefore  $r \sim O(\log N / \log(1 - \epsilon)^{-1})$ . The method is thus efficient.

In the case that the final state  $|\phi_f\rangle$  (before measurement) of the quantum computer is not a single product state, simply repeating the whole computation may not succeed since the state  $|\phi_f\rangle = \sum_u a_u |u\rangle$  can consist of a superposition of product states  $|u\rangle$  in which the number of terms in the superposition is exponentially large, as for example in Shor's algorithm. In this case the problem can be avoided by error-correction coding [17], using a repetition code in the measurement basis. The coding requires a repetition  $r$  of the same order as that given in the previous paragraph. The measurement of the computer's final state must be post-processed (by classical computation) to implement the error correction, after which the value  $u$  for one of the product states  $|u\rangle$  in  $|\phi_f\rangle$  can be deduced with high probability. The situation is now equivalent to having ideal measurements.

Large quantum computers will rely heavily on quantum error correction, applied repeatedly during the computation. Such fault-tolerant methods work best when the error syndromes can be measured, rather than handled by unitary quantum networks [18], since the non-trivial syndrome interpretation can then be carried out reliably by a classical computer. Hence it is important to ask whether  $\epsilon \sim 0.9$  renders the syndrome measurement too unreliable. A preliminary calculation using the methods of fault-tolerant syndrome extraction suggests that reliable computing can be achieved with this value of  $\epsilon$ , however this is a long calculation and will be presented elsewhere. It is a helpful feature that the zero syndrome, corresponding to no error, can be measured reliably.

### IV. SIDEBAND COOLING

So far we have established that quantum information processing experiments can be carried out in the ground state of ions such as  $^{40}\text{Ca}^+$ , with good final readout. The choice of state avoids the need for very highly stabilised lasers, and the Zeeman shelving method avoids the need for very efficient fluorescence detection, making such experiments significantly simpler than they would be otherwise. In this section we consider a further simplification, in which the need for very tight confinement in the ion trap is avoided.

Cooling of the motional degrees of freedom is one of the most difficult parts of quantum processing experiments. So far only single ions have been cooled to the quantum ground state of a trap, first in one dimension only [19], then in three dimensions [14]. Both experiments used *sideband cooling* [2,20] in the Lamb-Dicke limit. The Lamb-Dicke limit is the condition that the ion's motion is confined to a region small compared to the wavelength of the radiation under consideration. Since we are concerned with motional states near the ground state, of dimension  $a_0$ , this condition is

$\eta = 2\pi a_0/\lambda \ll 1$ . From equation (1), this places a constraint on the confinement provided by the trap, parametrised by the vibrational frequency along the  $z$  axis,  $\omega_z \gg E_R/\hbar$ . Single ions of the species we are considering have recoil frequencies  $E_R/\hbar$  in the region 6 to 30 kHz. Taking as an example a string of 10 ions we then require  $\omega_z \sim 2\pi \times 300$  kHz. To make the ions lie along a line, we need also  $\omega_z \ll \omega_{x,y} \sim 2\pi \times 3$  MHz. This degree of confinement is experimentally accessible, but requires either very small closely spaced trap electrodes, or careful high voltage r.f. design. In the former case the coupling between the ion motion and noise voltages in the electrodes is enhanced, and in the latter the r.f. voltages are liable to be more noisy. Therefore there are advantages in using a less tightly confining trap.

Another reason to avoid the Lamb-Dicke limit is that we need to drive vibrational-state-changing transitions ( $\Delta n = \pm 1$ , where  $n$  is the vibrational quantum number of the fundamental vibrational mode) using the laser radiation, for information processing in the ions. A small value of  $\eta$  is not optimal for this purpose because it increases the off-resonant excitation of  $\Delta n = 0$  transitions when  $\Delta n = \pm 1$  transition are driven. One could achieve the right conditions by first cooling to the ground state in the Lamb-Dicke limit, and then adiabatically opening the trap to  $\eta \sim 1$  before processing, but it is interesting to know if this adiabatic opening can be avoided. Furthermore, for  $\eta \simeq 1$  the cooling rate is higher than for  $\eta \ll 1$ , which permits a lower steady-state temperature to be reached since heating effects (other than the random recoil from spontaneous emission) typically scale more slowly with trap tightness than the sideband cooling rate.

Our approach to sideband cooling in the case of intermediate confinement, i.e.  $\eta \sim 1$ , is essentially to use both the first and a higher order sideband. Second sideband cooling in a standing wave was considered by de Matos Filho and Vogel [21], who showed that nonclassical motion can be obtained. We consider travelling wave radiation and are interested merely in obtaining low temperatures. Morigi *et al.* [22] recently discussed similar ideas to ours. They considered both  $\eta \simeq 1$  and the case  $\eta > 3$  where new features become apparent. We provide a more complete analysis of the region  $0 < \eta < 2$ . Other theoretical analyses [23–27] have been restricted to first-sideband cooling and the Lamb-Dicke limit, though typically with a more thorough description of the atom-light interaction or the trapping potential or both.

To describe the atom-laser interaction in the case of Raman transitions we adopt the standard theoretical device of considering the ground state manifold  $|M = \pm 1/2\rangle_g$  as an effective two-level atomic system [28,29]. This approximation is valid when the excited state populations are negligible. The effective Rabi frequency describing the atom-laser coupling is  $\Omega = \Omega_- \Omega_+ / 2\Delta$  where  $\Omega_{\pm}$  are the single-photon Rabi frequencies for excitation from  $|M = \pm 1/2\rangle_g$  to an excited state such as  $P_{1/2}$  or  $P_{3/2}$  (including Clebsch-Gordan coefficients), and  $\Delta \gg \Omega_{\pm}$  is the detuning from resonance of these transitions. The effective detuning is

$$\delta = \omega_{L-} - \omega_{L+} - \omega_{+-} - \frac{|\Omega_-|^2 - |\Omega_+|^2}{4\Delta}, \quad (14)$$

where  $\omega_{L\pm}$  are the laser frequencies and  $\omega_{+-} = \mu_B B/\hbar$  is the Zeeman splitting of the  $|M = \pm 1/2\rangle_g$  levels. The effective linewidth  $\Gamma$  is equal to the optical pumping rate from  $|M = +1/2\rangle_g$  to  $|M = -1/2\rangle_g$  caused by a  $\sigma^-$ -polarized laser resonant with a transition such as  $|M = +1/2\rangle_g \leftrightarrow |P_{1/2}, M = -1/2\rangle$ . This laser radiation is either very weak and continuously present, or moderately strong and pulsed on when a spontaneous transition is desired.

### A. First sideband cooling

We will discuss cooling of a single ion along one dimension. The generalisation to many ions and three dimensions is straightforward. Following Wineland and Itano [23], we will first account for the average change in total energy of the ion during one complete cycle of ‘excitation’ ( $|M = -1/2\rangle_g \rightarrow |M = +1/2\rangle_g$ ) followed by ‘spontaneous emission’ ( $|M = +1/2\rangle_g \rightarrow |M = -1/2\rangle_g$ ). The average is taken over many such cycles, in which the direction of spontaneously emitted photons varies randomly. The total state of the ion will be written  $|M = \pm 1/2\rangle_g |n\rangle$  where the  $\pm$  indicates the internal state, and the integer  $n \geq 0$  indicates the vibrational state (external energy eigenstate) of the ion in the trap.

Let

$$I_{fn} \equiv |\langle f | e^{i\mathbf{k}\cdot\mathbf{z}} | n \rangle|^2. \quad (15)$$

where  $z$  is the position of the centre of mass of the ion. This is the relative strength of the different sideband components in the ion-radiation field interaction [23]. Using  $\mathbf{k} \cdot \mathbf{z} = \eta(\hat{a}^\dagger + \hat{a})$  where  $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$  one obtains

$$\langle f | e^{i\mathbf{k}\cdot\mathbf{z}} | n \rangle = e^{-\eta^2/2} \sqrt{n!f!} (i\eta)^{|f-n|} \sum_{m=0}^{\min(n,f)} \frac{(-1)^m \eta^{2m}}{m!(m+|f-n|)!(\min(n,f)-m)!}. \quad (16)$$

The optical cross section for the ‘absorption’ transition  $|M = -1/2\rangle_g |n\rangle \rightarrow |M = +1/2\rangle_g |f\rangle$  is

$$\sigma_{fn}(\delta) = \sigma_0 I_{fn} \tilde{g}(\delta - (E_f - E_n)/\hbar) \quad (17)$$

where  $\tilde{g} = g\Gamma\pi/2$  is the lineshape function (5) normalised such that  $\tilde{g}(0) = 1$ . If the occupation probability of vibrational state  $n$  is  $P_n$ , then the rate of change in total energy of the ion during absorption, averaged over many cycles, is

$$\frac{dE_{\text{abs}}}{dt} = \sum_n P_n \sum_f (E_f - E_n) (I/\hbar\omega_{+-}) \sigma_{fn}, \quad (18)$$

where  $I$  is the effective laser intensity.

The average change in ion energy during spontaneous emission is

$$\begin{aligned} E_{\text{sp}} &= \int d\Omega_{\mathbf{k}} P(\mathbf{k}) \sum_n (E_n - E_f) I_{nf} \\ &= \int d\Omega_{\mathbf{k}} P(\mathbf{k}) \sum_n \langle f | e^{i\mathbf{k}\cdot\mathbf{z}} | n \rangle \langle n | [H^{\text{ext}}, e^{-i\mathbf{k}\cdot\mathbf{z}}] | f \rangle \\ &= \int d\Omega_{\mathbf{k}} P(\mathbf{k}) \langle f | \frac{\hbar^2 k^2}{2M} - \frac{\hbar \mathbf{k}}{M} \cdot \mathbf{p} | f \rangle \\ &= E_R. \end{aligned}$$

Note that since  $\langle f | \mathbf{p} | f \rangle = 0$  this answer is independent of the distribution  $P(\mathbf{k})$  of directions of spontaneous emission, which is counterintuitive. If, instead, the absorption and spontaneous emission are treated as a single scattering process (see eq. (27)), then the same result for the mean energy change is obtained as long as the distribution  $P(\mathbf{k})$  is symmetric about the origin, as it is in practice.

Combining the contributions of absorption and spontaneous emission, the mean rate of change of total energy, per cycle of absorption followed by spontaneous emission, is

$$\frac{d\langle E \rangle}{dt} = \frac{I\sigma_0}{\hbar\omega_{+-}} \sum_n P_n \sum_f (E_f - E_n + E_R) I_{fn} \tilde{g}(\delta - (E_f - E_n)/\hbar) \quad (19)$$

This equation was derived in [23] as part of a more general discussion. The above shows that its derivation can be simple and physically intuitive. It is valid to separate absorption and spontaneous emission because typically the experimental procedure is to switch on the Raman lasers and optical pumping laser at separate times [19,14]. However, later we will not separate them but treat the emission and absorption as a photon scattering process.

Although we are interested in the case  $\eta \sim 1$ , it is instructive first to examine the Lamb-Dicke limit  $\eta \ll 1$ . This is done by a series expansion of eq. (19) in powers of  $\eta^2$ . The expansion of  $I_{fn}$  (eq. (15)) to order  $\eta^4$  is

$$\begin{aligned} I_{fn} &= \delta_{f,n} \left( 1 - \eta^2(2n+1) + \frac{\eta^4}{2}(3n^2 + 3n + 1) \right) \\ &\quad + \delta_{f,n-1} \eta^2 n (1 - \eta^2 n) + \delta_{f,n+1} \eta^2 (n+1) (1 - \eta^2(n+1)) \\ &\quad + \delta_{f,n-2} \frac{\eta^4}{4} n(n-1) + \delta_{f,n+2} \frac{\eta^4}{4} (n+1)(n+2). \end{aligned}$$

This can be obtained either from eq. (16) or by expanding the operator  $e^{i\mathbf{k}\cdot\mathbf{z}}$  in eq. (15). The terms in eq. (19) up to order  $\eta^2$  are

$$\begin{aligned} \frac{d\langle E \rangle}{dt} &= \frac{I\sigma_0 E_R}{\hbar\omega_{+-}} \sum_n P_n \{ (\tilde{g}_m + \tilde{g}_{m+1}) + \eta^2(\tilde{g}_{m+2} - \tilde{g}_m) \\ &\quad + n((\tilde{g}_{m+1} - \tilde{g}_{m-1}) + \eta^2(\tilde{g}_{m-1} - \tilde{g}_{m+1} + \tilde{g}_{m-2}/2 - 2\tilde{g}_m + 3\tilde{g}_{m+2}/2)) \\ &\quad + n^2 \eta^2 (-\tilde{g}_{m-2}/2 - \tilde{g}_{m+1} + \tilde{g}_{m-1} + \tilde{g}_{m+2}/2) \}, \end{aligned} \quad (20)$$

where  $m = 1, 2, \dots$  indicates the sideband chosen for cooling, and  $\tilde{g}_m \equiv \tilde{g}(m\omega_z) \equiv \Gamma^2/(4m^2\omega_z^2 + \Gamma^2)$ .

For first sideband cooling,  $m = 1$ , in the limit of well-resolved sidebands we ignore terms of order  $\Gamma^2/\omega_z^2$  compared to 1 to obtain, for the steady state distribution ( $d\langle E \rangle/dt = 0$ ),

$$\tilde{g}_1 + \tilde{g}_2 + \eta^2(\tilde{g}_3 - \tilde{g}_1) + \langle n \rangle(-1 + \eta^2) + \langle n^2 \rangle \eta^2 = 0 \quad (21)$$

In the Lamb-Dicke limit one finds that the rate of driving of population from one level to another is independent of the level number  $n$ , so the principle of detailed balance leads to the thermal distribution defined by  $P_{n+1}/P_n = e^{-\hbar\omega_z/k_B T} \equiv s$ . The populations are then

$$P_n = (1 - s)s^n \quad (22)$$

and the mean and mean square quantum number are

$$\langle n \rangle = \frac{s}{1 - s}, \quad \langle n^2 \rangle = \frac{s(1 + s)}{(1 - s)^2}. \quad (23)$$

from which  $\langle n^2 \rangle = 2\langle n \rangle^2 + \langle n \rangle$ . Substituting this in (21) and solving for  $n$ , ignoring terms of order  $\eta^4$  or  $\Gamma^4/\omega_z^4$ , we obtain

$$\langle n \rangle \simeq \frac{\tilde{g}_1 + \tilde{g}_2 - \eta^2(\tilde{g}_1 - \tilde{g}_3)}{1 - 2\eta^2} \simeq \frac{(5 - 32\eta^2/9)\Gamma^2}{16(1 - 2\eta^2)\omega_z^2} \quad (24)$$

This result is precise in the Lamb-Dicke limit with well-resolved sidebands. Figure 5 shows  $1 - P_0$  as a function of  $\eta^2$  as given by (24) (dashed line) and by a more exact treatment (full line) which will be described shortly. It is seen that (24) indicates the right qualitative behaviour for moderate values of  $\eta$ , though it is not precise, owing to a slight departure from a thermal population distribution in the more exact treatment. The main conclusion is that first sideband cooling works very well for values of the Lamb-Dicke parameter up to  $\eta \sim 0.6$ . It is noteworthy that the effective temperature given by  $k_B T = \hbar\omega_z/\ln(P_0/P_1)$  is approximately equal to the recoil limit  $k_B T = E_R$  when  $\eta = 0.5$ ,  $\Gamma = 0.2\omega_z$ , and is sub-recoil for smaller linewidths.

## B. Second sideband cooling

For second sideband cooling,  $m = 2$ , eq. (20) can be usefully simplified by neglecting terms of order  $O(\eta^2\Gamma^2/\omega_z^2)$ . In other words, we make immediately the assumption of well-resolved sidebands. The steady state of eq. (20) is then given by

$$\langle n^2 \rangle \frac{\eta^2}{2} + \langle n \rangle (\tilde{g}_1 - \tilde{g}_3 - \frac{\eta^2}{2}) - (\tilde{g}_2 + \tilde{g}_3) = 0. \quad (25)$$

There are two interesting regimes in which this result can be examined. First, for a very tight trap with  $\eta^2 \ll \Gamma^2/\omega_z^2$ , the  $\eta^2$  terms are negligible and we have

$$\langle n \rangle = \frac{\tilde{g}_2 + \tilde{g}_3}{\tilde{g}_1 - \tilde{g}_3} \simeq \frac{13}{32}. \quad (26)$$

This can be understood as the condition that all the population is driven to the ground and first excited vibrational levels, but that these are almost equally populated since the laser drives off-resonant transitions from each to the other at almost equal rates.

Secondly, let us consider a trap that is less tight though the sidebands are still well resolved:  $\Gamma^2/\omega^2 \ll \eta^2 \ll 1$ . Since the population in levels  $n > 1$  is optically pumped efficiently towards  $n = 0, 1$ , most of the population is in  $P_0$  and  $P_1$ , and therefore  $\langle n^2 \rangle \simeq P_1 \simeq \langle n \rangle$ . Using this to express  $\langle n^2 \rangle$  in terms of  $\langle n \rangle$  in eq. (25), the  $\eta^2$  terms cancel, so once again the mean quantum number is given by (26). This reasoning is valid as long as  $\langle n^2 \rangle - \langle n \rangle \ll \tilde{g}_2 + \tilde{g}_3$ .

In summary, cooling on the second sideband alone does not allow the ground state population to reach a value close to 1. Equation (26) is nevertheless useful as a further test (in addition to (24)) on the numerical treatment to be described in the next section.



### C. Cooling on two sidebands

We now turn our attention to  $\eta > 0.6$ . For single-frequency laser radiation on the  $m$ th sideband, the populations  $P_n$  can be deduced from a set of rate equations, in which the rate of transfer of population from level  $n$  to level  $f$  is given by

$$\Gamma_{fn}(m) = \frac{\Omega^2}{\Gamma} \int_{-1}^1 du N(u) \left| \sum_{j=0}^{\infty} \frac{\langle f | e^{-ikzu} | j \rangle \langle j | e^{ik \cdot \mathbf{z}} | n \rangle \Gamma/2}{\omega_z(-m-j+n) + i\Gamma/2} \right|^2, \quad (27)$$

where  $N(u)$  is the angular distribution of spontaneous emission. Note that we are now treating the atom-laser interaction as a scattering process (still in the low intensity limit  $\Omega \ll \Gamma, \omega_z$ ). If one wished to calculate the case that absorption and emission are separate processes (i.e. a driven transition followed by optical pumping), one would take the modulus square of each term in the sum over  $j$ , instead of the modulus square of the whole sum. We find however that the two cases give almost identical results, so we will restrict our discussion to the form (27).

For dipole radiation in three dimensions one typically uses  $N(u) = \frac{3}{8}(1+u^2)$ . To remain consistent with the one-dimensional approach we have adopted, we take instead  $N(u) = (\delta(u+1) + \delta(u-1))/2$ , which corresponds physically to spontaneously emitted photons propagating along the  $z$  axis. We find that this correctly reproduces the Lamb-Dicke limit behaviour described in the previous sections.

When  $\eta > 0.6$ , single frequency radiation will not cool the ion to the motional ground state. We therefore consider the next simplest case, namely two laser frequencies, one tuned to the first sideband, the other to the sideband  $m > 1$  (both are red sidebands, i.e. below the carrier frequency). To allow two laser frequencies in the population rate equations (27), we perform an incoherent sum, giving for the rate of transfer from  $n$  to  $f$

$$\Gamma_{fn} = \Gamma_{fn}(m) + \alpha \Gamma_{fn}(1). \quad (28)$$

This corresponds physically to the case that the radiation fields at the two frequencies are not simultaneously present, but pulsed on one after the other, and we consider the behaviour time-averaged over such switching. The parameter  $\alpha$  is the ratio of the two laser intensities.

The general physical picture of the behaviour is that the laser on the  $m$ th sideband pumps the population towards the lowest  $m$  vibrational levels, and cools overall, while the relatively low intensity first sideband laser has a small heating effect overall, but efficiently pumps the population from the lowest levels into the ground state. The rate of this pumping into the ground state is approximately  $\alpha(\Omega^2/\Gamma)I_{00}I_{01} = \alpha(\Omega^2/\Gamma)\eta^2 e^{-2\eta^2}$ .

In figures 6a and 6b we present the result of numerically extracting the steady-state populations, for a set of population rate equations using the rates (28). The figures show the value of  $1 - P_0$  for  $\eta^2$  in the range 0 to 3, for two values of the linewidth, and for  $m = 1$  to 4. The numerical calculations used a truncated set of levels, from the ground state  $n = 0$  to  $n = n_{\max}$ . Even though the population is almost all in the low levels, it was found that a large number of levels ( $n_{\max} = 100$ ) had to be included in order to calculate  $1 - P_0$ . This is because for  $\eta > 0.6$  the population distribution has a long oscillating ‘tail’ which falls off as a low power of  $n$ . As a result, the results in figs 6a and 6b have only approximately 5% accuracy on the parts of the curves which rise steeply as a function of  $\eta^2$ .

We found by numerical experiment that the ground state population does not depend strongly on the laser intensity ratio  $\alpha$ , and that the optimal value is approximately given by  $\alpha = 1/(3\eta^2)$ . This choice was adopted for all the results presented here.

The main conclusions from figure 6 are as follows. We find that in the limit of well-resolved sidebands,  $1 - P_0$  is roughly proportional to  $\Gamma^2/\omega_z^2$  for the whole range of  $\eta$  values shown. The best sideband to choose, in order to maximise  $P_0$  at given  $\eta$ , is higher than one might expect. A naive argument would suggest that since resonant absorption results in a kinetic energy decrease by  $m\omega_z$ , and spontaneous emission results on average in a kinetic energy increase of  $E_R = \eta^2\omega_z$ , the best choice is  $m$  equal to the smallest integer larger than  $\eta^2$ , ignoring the small heating effect of the first sideband laser. In fact we find the optimum  $m$  is the smallest integer larger than approximately  $2\eta^2 + 0.5$ .

The poor performance shown in fig. 6 around  $\eta^2 = 1.27$  and  $\eta^2 = 2$  is due to zeroes in  $I_{fn}$ , namely  $I_{23}(\eta^2 = 3 - \sqrt{3}) = 0$  and  $I_{12}(\eta^2 = 2) = 0$ . The first makes the depopulation of level 3 inefficient when  $m > 3$ , the second makes the depopulation of level 2 inefficient when  $m > 2$ . The structure in the  $m = 3$  curve at  $\eta^2 \simeq 1.5, 1.3, 1.2, \dots$  is due to nearly coincident zeroes of  $I_{n-3,n}$  and  $I_{n-1,n}$ , causing population to accumulate at  $n = 8, 9, 10, \dots$  respectively. Such coincidences only happen when both sidebands are odd, or both even, which is why similar structure does not appear in fig. 6 for  $m = 2, 4$ .

The method of cooling on a pair of sidebands will not work for large  $\eta$ , because the first sideband laser cannot efficiently drive transitions from level 1 to 0. This was noted by Morigi *et al.* [22]. We can roughly estimate the maximum  $\eta$  for which cooling to the ground state is possible as follows.

The equations (27) are too complicated to solve analytically when  $\eta$  is not small. In order to make rough estimates, we make several approximations. First approximate  $\Gamma_{fn}$  by ignoring the distribution  $N(u)$  and assuming the interference terms in the sum roughly cancel, leaving

$$\Gamma_{fn} \simeq \frac{\Omega^2}{\Gamma} \sum_{j=0}^{\infty} I_{fj} I_{jn} (\tilde{g}_{n-j-m} + \alpha \tilde{g}_{n-j-1}). \quad (29)$$

We are interested in the case of low temperature, where most of the population is in the low-lying levels. In the limit that most of the population is contained in  $P_0$  and  $P_1$ , the ratio  $r \equiv P_0/P_1$  can be estimated to be  $r \simeq \Gamma_{01}/\Gamma_{10}$ . If  $r$  is large then  $P_0 \simeq 1$ , so cooling to the ground state has been achieved, and

$$P_0 \simeq \frac{\Gamma_{01}}{\Gamma_{01} + \Gamma_{10}}. \quad (30)$$

We now examine the ratio

$$\frac{\Gamma_{01}}{\Gamma_{10}} \simeq \frac{\sum_{j=0}^{\infty} I_{0j} I_{j1} (\tilde{g}_{m+j-1} + \alpha \tilde{g}_j)}{\sum_{j=0}^{\infty} I_{1j} I_{j0} (\tilde{g}_{m+j} + \alpha \tilde{g}_{j+1})}. \quad (31)$$

Since  $m > 1$ , the only resonant term in the two sums is the  $\tilde{g}_j$  term in the numerator for  $j = 0$ . Therefore the ratio may be written, in the limit  $\Gamma^2 \ll \omega_z^2$ ,

$$\frac{\Gamma_{01}}{\Gamma_{10}} \simeq \frac{I_{00} I_{01} + \beta}{\beta} = \frac{e^{-2\eta^2} \eta^2 + \beta}{\beta} \quad (32)$$

where  $\beta$  is  $O(\Gamma^2/\omega_z^2)$  and we used the expression for  $I_{00}$  and  $I_{01}$  given by (16). The factor  $\alpha$  cancels provided it is larger than  $\sim \Gamma^2/\omega_z^2$ . We estimate  $\beta$  as follows. By definition,

$$\begin{aligned} \beta &\simeq \sum_{j=0}^{\infty} I_{1j} I_{j0} \tilde{g}_{j+1} \\ &\simeq \frac{\Gamma^2}{4\omega_z^2} \sum_{j=0}^{\infty} e^{-\eta^2} \frac{\eta^{2j}}{j!} e^{-\eta^2} \frac{\eta^{2(j-1)}}{j!} (j - \eta^2)^2 \frac{1}{(j+1)^2} \\ &\simeq \frac{\Gamma^2}{4\omega_z^2} e^{-2\eta^2} \eta^{-2} \sum_{j=0}^{\infty} \frac{(\eta^2)^{2j}}{(j!)^2} \end{aligned} \quad (33)$$

where we used (16) for  $I_{1j}$  and  $I_{j0}$ . The last step used  $(j - \eta^2)^2 \simeq (j+1)^2$  which is true for large  $j$  and enables the sum to be performed. We find by numerical calculation that this overestimates  $\beta$  by a factor  $\sim \eta^2$ . The sum in (33) is equal to the Bessel function  $I_0(2\eta^2)$  whose value for  $2\eta^2 > 2$  is

$$I_0(2\eta^2) = \frac{e^{2\eta^2}}{\eta\sqrt{4\pi}} (1 + O((4\eta)^{-2})). \quad (34)$$

Substituting this in (33) we obtain

$$\beta \sim \frac{\Gamma^2}{4\omega_z^2} \frac{1}{2\sqrt{\pi}} \eta^{-3}. \quad (35)$$

The most important feature of this calculation is that whereas  $I_{00} I_{01}$  falls exponentially with  $\eta^2$ , we find that  $\beta$  does not, therefore it is impossible to achieve  $I_{00} I_{01} > \beta$  (and hence a large ground state population) once  $\eta$  exceeds a value that depends insensitively on  $\Gamma/\omega_z$ . In other words the optical pumping from  $n = 1$  to  $n = 0$  becomes exponentially inefficient. We can estimate the limiting value of  $\eta$  by setting  $I_{00} I_{01} = \beta$ , giving

$$\eta^2 \sim \ln(2\omega_z/\Gamma) + \frac{7}{4} \ln(\eta^2) + 2. \quad (36)$$

where the power  $\eta^{-3}$  in equation (35) has been replaced by  $\eta^{-5}$  in agreement with our numerical calculation. Equation (36) gives a reasonable estimate of the maximum value of  $\eta$  that will permit accumulation of population in the ground

state by cooling on the first and a higher sideband. For  $\Gamma = 0.1\omega_z$ , the limiting value is  $\eta \simeq 3$ , in agreement with the results of [22].

In conclusion, let us comment on the best choice of experimental parameters implied by this study. Our electron-shelving measurement technique has as its major advantage the possibility of a high signal-to-noise ratio, since fluorescence can be driven for a long time before an unwanted transition makes the signal ambiguous. This renders information processing experiments in the ground state of  $^{40}\text{Ca}^+$  and similar ions feasible. Our study of sideband cooling has indicated the best choice of laser parameters in order to achieve a large population of the ground state of motion. In order to retain the simplicity of only cooling on a single sideband, combined with a not too great suppression of vibrational state-changing transitions, the choice  $\eta = 0.5$  is suitable.

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## FIGURE CAPTIONS

Fig. 1. Low lying levels in  $^{40}\text{Ca}^+$ . The Zeeman structure of the levels is shown, though the level spacings are not to scale. The wavelengths and inverse spontaneous rates of the electric-dipole allowed transitions are given. The circled numerals indicate the numbering of the levels adopted for equations (2) to (11).

Fig. 2. Shelving efficiency  $\epsilon$  as a function of magnetic field in Tesla, for two values of the Rabi frequency of the shelving pulse: dashed line  $\Omega_{15} = \Gamma_{15}$ ; full line  $\Omega_{15} = \Gamma_{15}/4$ . The shelving pulse duration was chosen to maximise  $\epsilon$  for each value of magnetic field and Rabi frequency.

Fig. 3. Shelving efficiency as a function of shelving pulse duration, for  $\Omega_{15} = \Gamma_{15}$ ,  $B = 0.01$  Tesla. Full line: solution of full rate equations; dashed line: equation (12).

Fig. 4. Steady state population of  $P_{1/2}$  level,  $N_{-1/2}^{(4)} + N_{+1/2}^{(4)}$ , during the detection phase, as a function of magnetic field. The observed fluorescence signal is proportional to this population. The two laser beam intensities are set to give equal Rabi frequencies,  $\Omega \equiv \Omega_{14} = \Omega_{24}$ . Reading from the upper (full) curve to the lower (dash-dotted) curve, this Rabi frequency is  $\Omega = 10\Gamma_4$ ,  $5\Gamma_4$ ,  $2\Gamma_4$ ,  $\Gamma_4$  respectively.

Fig. 5. First sideband cooling. The figure shows the difference between the ground state population  $P_0$  and 1, as a function of  $\eta^2$  for first sideband cooling with  $\Gamma = 0.1\omega_z$ . Full line: numerical solution of population rate equations using rates  $\Gamma_{fn}(1)$  (eq. (27)); dashed line: analytic solution (24) in Lamb-Dicke limit assuming a thermal population distribution, giving  $1 - P_0 = s \simeq \langle n \rangle$ .

Fig. 6. Cooling on two sidebands. The curves show the difference between  $P_0$  and 1, as a function of  $\eta^2$ , for cooling on the first sideband and the  $m$ th, where  $m = 1$  (full line), 2 (dashes), 3 (dots), and 4 (dash-dot), with  $\alpha = 1/(3\eta^2)$  which roughly maximises  $P_0$ . (a):  $\Gamma = 0.1\omega_z$ ; (b):  $\Gamma = 0.2\omega_z$ . The curves are obtained by numerical solution of the steady-state population rate equations with 100 vibrational levels included in the calculation.













